Kelly Chao
Limit Definition, Derivative Rules, Linear Approximation, Tangents
12/15/19

Limit Definition

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Write this down 3 times:

1.

2.

3.

$f(x)$ is a function where x is the input variable.

$f(x+h)$ is another function where $x+h$ is the input variable.

h is an infinitesimally small value that when it approaches 0, we can find the infinitesimally small change between the two functions.

$f(x+h) - f(x)$ is the change between the two functions when h approaches 0.

The notation for h approaching 0 is $h \to 0$

When we take the limit of $h \to 0$ we give it this notation: $\lim_{h \to 0}$.

We can't leave that little box open, though, without finding the difference between the two functions, one with h added to x, and the other where the function input is only x.

Essentially the problem boils down to finding the change of y/x, where $f(x+h) - f(x) = y_2 - y_1$ and $h = x_2 - x_1$, where $x_2 = x + h$.

This means that we are finding the infinitesimal rate of change of the slope, in other words: $\left(\frac{\Delta y}{\Delta x}\right)$. We don't normally use this specific notation because it can get confusing. That is why we use

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Try to write down the equivalence of each variable.

Gather each variable and write them first in non-calculus notation, then in calculus notation.

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Derivative Rules

There are 7 general formulas for different types of derivatives. The way to go about solving these is to look at the problem and see if they roughly fit any of the 7 formulas. The 7 formulas can be applied to almost any problem. Sometimes, you will have to combine some of the formulas, but don't worry, as that is a very straightforward process usually done with addition, or multiplication. The more formulas you combine, the longer the process of solving for the derivative, but it is all very algorithmic. As long as you keep to recognizing the formula structure within the expressions and solve them how you would solve any simple variation of the problem, you are golden. We will go over what kind of combinations you will see, and how you can spot which formula, or formulas to apply to the expression.

1. Constant Rule

$$\frac{d}{dx}(c) = 0$$

The derivative of a lone constant is always zero.

2. Constant Multiple Rule

$$\frac{d}{dx}c * f(x) = c\frac{d}{dx}f(x)$$

The derivative of a [constant multiplied by a function] is the same as the constant multiplied by the [derivative of the function]. Essentially, with any derivative that has a constant in it, you can pull it out to the front and leave it there. The Constant Multiple Rule is one of the reasons $\left(\frac{d}{dx}\right)$ is a linear operator.

3. Sum Rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The derivative of the addition of 2 (or more) functions is equal to taking the derivative of each function first and THEN adding them together. The derivative operator $(\frac{d}{dx})$ is a linear operator because of the Sum Rule as well. For an operator to be linear, it must comply with the Constant Multiple Rule AND the Sum Rule, not either or, but both. Most of the math you have learned so far and will learn in a few years to a decade is built on this concept. It recurs "numerous" times. Pun intended.

4. Difference Rule

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Same idea as the sum rule, just with a negative sign.

5. Product Rule

$$\frac{d}{dx}[f(x) * g(x)] = \left[\frac{d}{dx}f(x) * g(x)\right] + \left[\frac{d}{dx}g(x) * f(x)\right]$$

The product rule is a combination of the sum rule and the constant multiple rule. The idea is that when you take the derivative of one function, the other function has to stay the same and be multiplied by the differentiated function. You do this for however many functions you may have attached together with multiplication, and you add them altogether in the end.

6. Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left\{\left[\frac{d}{dx}f(x) * g(x)\right] - \left[\frac{d}{dx}g(x) * f(x)\right]\right\}}{[g(x)^2]}$$

As you can see this is a little bit more complicated of a formula, however, there are some similarities between the other rules we have visited already. For one, the numerator is essentially the same as the product rule, except there is a negative sign in between. Finally, the denominator is just the original second function squared.

7. Chain Rule

$$\frac{d}{dx}f\big(g(x)\big) = \frac{d}{dx}f\big(g(x)\big) * \frac{d}{dx}g(x)$$

When you have the unfortunate case of inputting a function into another function, you first must take the derivative of that function as a whole, then you multiply it by the derivative of the inner function. The inner function in this case is g(x) while the outer function is f(g(x)).

How can we combine all of these formulas together? That seems so difficult. Well for the most part they are already variations of each other.

Chain Rule + Product Rule

$$\frac{d}{dx}\left[f\big(g(x)\big) * g\big(f(x)\big)\right] = \{\frac{d}{dx}[f(g(x))] * g\big(f(x)\big)\} + \left\{\frac{d}{dx}[g\big(f(x)\big)] * f\big(g(x)\big)\right\}$$
$$= \left\{\frac{d}{dx}[f\big(g(x)\big)] * \frac{d}{dx}[g(x)] * g\big(f(x)\big)\right\} + \left\{\frac{d}{dx}[g\big(f(x)\big)] * \frac{d}{dx}[f(x)] * f\big(g(x)\big)\right\}$$

It can get confusing to do this but as long as you're able to keep track of everything and remember the patterns, you will be okay. Try highlighting all the patterns you can find within this. Can you see the chain rule? Can you see where the product rule was placed?

Now try it on your own

Chain Rule + Difference Rule

$$\frac{d}{dx}\left[f\big(g(x)\big) - g\big(f(x)\big)\right] =$$

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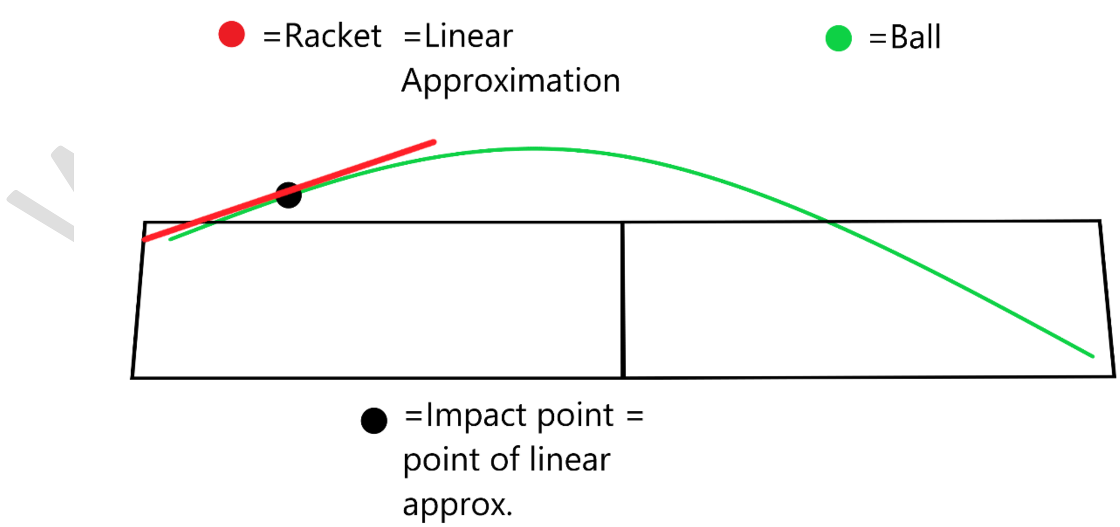
Difference Rule + Product Rule

$$\frac{d}{dx}\{[f(x) * g(x)] - [h(x) * s(x)]\} =$$

See if you can come up with your own variation and solve it.

Linear Approximation

Every time you hit the ball in tennis, you're essentially linearly approximating where you want the ball to go without being consciously aware of it. Let's say you're in the left corner of the court hitting an inside out



forehand cross-court. The line that you follow with your racket as you're spinning the ball cross-court is the linear approximation line for the ball that will follow a parabolic curve down the court. There is a relationship between the two trajectories. What line does your racket follow so you can hit the ball right into the corner cross-court of the other side of the court?

This will help you recognize when to use linear approximations. You will have some sort of curve, and you will be asked to find the slope of a point on that curve. You've most likely done this before in middle school, or early high school, but in calculus, we add differentiation into the equation to make solving equations simpler from the algebraic jumble you probably have come across before.

Formula:

$$L(x) = Linear\ Approximation\ of\ f(x)\ at\ the\ point\ a$$
$$L(x) = f(a) - \frac{d}{dx}[f(a)] * (x - a)$$
$$f(x) = ball\ trajectory$$
$$a = racket\ and\ ball\ impact\ point$$
$$\frac{d}{dx}[f(x)] = racket\ trajectory$$

If you look at the general structure of this equation, you may say it is reminiscent of point slope form of y=b+mx.

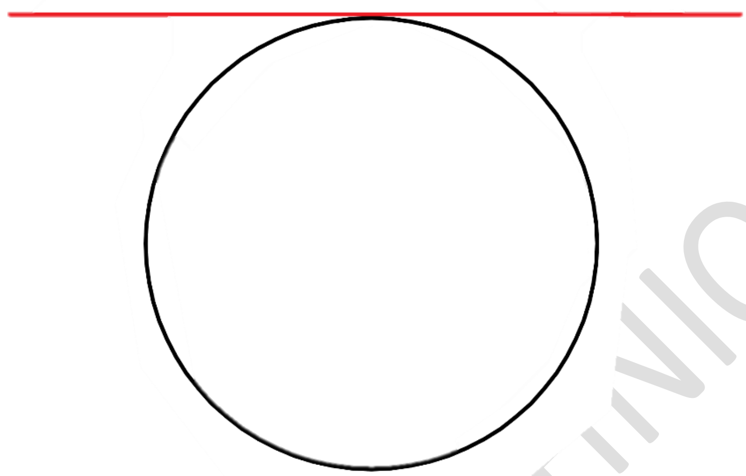
See if you can take the point slope form variables and translate them into calculus variables.

Can you put the linear approximation variables into point slope form? What happens when you solve it algebraically?

Tangents

Linear approximation is the formula and process used to find the tangent line. The red line in the tennis picture is the tangent line. To find the tangent line, you need to linearly approximate and to linearly approximate you simply use the formula in the previous section that you derived.

Let's say you're going to your favorite boba tea place and you get your favorite Taro boba tea with a 50% discount. Good day, right? If you ever looked at each tapioca you might've noticed that they're spherical. For simplicity's sake let's assume they are circle shaped. Let's find the tangent line to the very top point of that circle.



First look at this, what is the slope of the tangent line?

If you remember your unit circle, what point is the tangent line on if it's at the very top?

What equation would you use to find the tangent line at the top of the circle?

Extra Practice Problems:

Plug in $f(x) = x^2 + \sin(x)$ into the limit definition. Think about where you will be putting h.

Find the derivative of:

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$$\frac{x^2}{4\sin(x)}$$

$$4x * cos(x) - \tan(x) * \cos(x)$$

$$\sec(\tan(\sec(x)))$$

Find the tangent line of:

$$\frac{x^2}{4\sin(x)}$$

$$4x * cos(x) - \tan(x) * \cos(x)$$

$$\sec(\tan(\sec(x)))$$

At $a = \frac{\pi}{4}$

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Comments:

Hey, Kelly! Really good job last video session on keeping track of your homework and being prepared for the video lesson with questions and topics. I'm happy you took the time beforehand to look over the topics to try and figure them out on your own, it shows you are diligent and thoughtful. I hope this study pamphlet helped you and was able to answer all your questions. I added some extra questions in there for you to answer and think about so that you're prepared for new material and have a mental image to revert to when you're studying for your test next week. Feel free to print this out and highlight things, send it to your friends or show your teacher if you're stuck. All in all, you're doing a great job!

Sincerely,

Anika